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Twisted supermultiplets

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Abstract. We investigate the topological effects of non simply connected space–time on supergravity with matter supermultiplets. Topological sectors are typically labelled by one or a pair of elements from the group of homomorphisms of $\pi_1(M)$ into Z_2 and Lorentz invariant vacuum states are defined by analogy with the θ-vacua of Yang–Mills theory.

1. Introduction

Much recent work on quantum gravity has involved topological properties of the space–time manifold. For example, a series of papers by one of us (CJI) and S J Avis has studied the quantum theory of fields propagating on a manifold M that is not simply connected. This topological property manifests itself in the existence of non trivial line bundles on M whose cross-sections may be regarded as a generalisation of the concept of a scalar field. It also leads to the existence of inequivalent spin connections. Both features give significant quantum field theoretic effects such as different S-matrices, self energies and vacuum polarisations in the different topological sectors.

One striking property is that the set of real line bundles on M and the set of inequivalent spin structures are both labelled by the same cohomology group $H^1(M; Z_2)$—the group of homomorphisms of the fundamental group $\pi_1(M)$ into Z_2. This suggests that theories involving both spinor and scalar fields may be particularly interesting from this point of view and a natural example is afforded by supersymmetric multiplets.

The present paper is therefore concerned with the study of supersymmetric field theories in a general non simply connected space–time (for a discussion when the space–time is metrically flat see Avis 1979). The purpose is to see if the boson and fermion members of a multiplet can be simultaneously twisted in a way which is compatible with supersymmetry. One anticipates that as in the case of a single spinor (Avis and Isham 1979) this effect will lead to a classification of vacuum states analogous to the θ-vacuum introduced in Yang–Mills instanton physics by Jackiw and Rebbi (1976) and Callan *et al* (1976). It must be emphasised that this is not just an empty mathematical exercise. There is in general no canonical way of selecting a particular spin connection on a non simply connected space–time and the whole set must be considered.

In § 2 we briefly review the manner in which twisted fields arise, and show that the concept of a twisted spin connection is equivalent to a twisted spinor field. We also discuss the special role played by Majorana spinors. In § 3 we investigate the assignment of topological twists to supergravity itself and to the scalar and Maxwell multiplets coupled to the gravity theory. The paper concludes with a short discussion.

We shall assume throughout that space–time is a paracompact pseudo-Riemannian manifold which is both space and time orientable and which admits spinors (and is hence parallelisable, Geroch 1968). Most of the discussion has a fairly obvious analogue in the Riemannian case provided that the extra topological complexity introduced by the possible non-triviality of the tangent bundle is allowed for.

2. Twisted scalar and spinor fields

2.1.

Let us briefly review the way in which twisted scalar fields and twisted spin connections arise. A normal real scalar field is a function from the space–time manifold M to the real numbers or, equivalently, a cross-section of the trivial real line bundle over M. If the topology of M is sufficiently complicated there may exist non trivial line bundles and a twisted scalar field is defined to be a cross-section of one of these (Isham 1978a). Such a bundle is associated with a principal $\mathrm{O}(1) \approx Z_2$ bundle and the set of these is classified by elements of the cohomology group $H^1(M; Z_2)$. Hence twisted real scalars can only exist if the space–time is not simply connected.

There are various ways of representing cross-sections. Since a vector bundle X is locally trivial, there exist coverings $\{U_\alpha\}$ of M such that X restricted to U_α is in product form. Thus there are maps h_α from $U_\alpha \times \mathbb{R}$ into $\tilde{\pi}^{-1}(U_\alpha)$ ($\tilde{\pi}$ is the projection map) and any cross-section ϕ possesses local representatives $\phi_{(\alpha)}$ mapping U_α into $\mathbb{R}$ and satisfying

$$\phi(x) = h_\alpha(x, \phi_{(\alpha)}(x)). \tag{2.1}$$

If $U_\alpha \cap U_\beta \neq \phi$ then $\phi_{(\alpha)}$ and $\phi_{(\beta)}$ are related on $U_\alpha \cap U_\beta$ by

$$\phi_{(\alpha)}(x) = g_{\alpha\beta}(x)\phi_{(\beta)}(x) \tag{2.2}$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{O}(1) = \{1, -1\}$ are the structure functions of the bundle.

Alternatively each Z_2-bundle over M is a double covering $\hat{M}$ of M with a projection map π and cross-sections of an associated line bundle are in bijective correspondence with functions $\tilde{\phi}$ from $\hat{M}$ into $\mathbb{R}$ satisfying

$$\tilde{\phi}(pa) = -\tilde{\phi}(p), \qquad \tilde{\phi}(pe) = \tilde{\phi}(p) \tag{2.3}$$

where e and a are respectively the identity and generator of $Z_2 \approx \mathrm{O}(1)$. The precise connection is

$$\phi(x) = [p, \tilde{\phi}(p)] \qquad \text{any } p \text{ in } \pi^{-1}(x) \tag{2.4}$$

where $[p, \nu]$ denotes the equivalence class of $(p, \nu) \in \hat{M} \times R$ defined in the setting up of the associated line bundle (Husemoller 1966). The second definition is related to the automorphic function approach exploited by Banach and Dowker (1978, 1979a,b), Banach (1979) in which fields are defined on the universal covering space of M.

Both classical and quantum equations of motion may be defined for twisted fields and a number of calculations illustrating the effects of these topologically distinct configurations have been performed (Isham 1978a, Avis and Isham 1978, Banach and

Dowker 1978, 1979a, b, Banach 1979, DeWitt *et al* 1979, Ford 1979, 1980a,b, Toms 1979a, b, Unwin 1979, 1980, Kennedy *et al* 1979, Unwin and Critchley 1979).

2.2.

A simple way of seeing the existence of twisted spin connections is as follows (Isham 1978b, Avis and Isham 1979). The Dirac Lagrangian for a spinor field in a gravitational background is

$$\mathscr{L}(e, \psi) = \det(e)\{\tfrac{1}{2}(\bar{\psi}\gamma_a \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma_a \psi)e^{a\mu} - m\bar{\psi}\psi\} \tag{2.5}$$

where

$$\nabla_\mu \psi = (\partial_\mu + \omega_\mu)\psi, \qquad \nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi}\omega_\mu \tag{2.6}$$

$$\omega_\mu = \tfrac{1}{2}\omega_{\mu ab}\sigma^{ab}, \qquad \sigma^{ab} = \tfrac{1}{4}[\gamma^a, \gamma^b] \tag{2.7}$$

$$\omega_{\mu ab} = \{{}_{\mu}{}^{\beta}{}_{\alpha}\} e_a{}^{\alpha} e_{b\beta} + e_{b\alpha} e_a{}^{\alpha}{}_{,\mu} \tag{2.8}$$

and $\{{}_{\mu}{}^{\beta}{}_{\alpha}\}$ and $e_{a\mu}$ are respectively the Christoffel symbol and vierbein.

Now subject the vierbein to a gauge transformation

$$e_{a\mu} \rightarrow e'_{a\mu} = e_{b\mu}\Omega^b{}_a \tag{2.9}$$

where Ω is a smooth map from M into the connected component of the Lorentz group $\mathscr{L}_\uparrow$. The corresponding transformation of the spin connection is

$$\omega_{\mu ab} \rightarrow \omega^{\Omega}_{\mu ab} = \omega_{\mu cd}\Omega^c{}_a\Omega^d{}_b + \Omega^c{}_b\Omega_{ca,\mu}. \tag{2.10}$$

If there exists a covering map S from M into $SL(2, \mathbb{C})$ such that

$$\Lambda(S(x)) = \Omega(x) \tag{2.11}$$

(Λ is the two-to-one homomorphism of $SL(2, \mathbb{C})$ onto $\mathscr{L}_\uparrow$) then (2.10) can be rewritten in matrix form as

$$\omega_\mu \rightarrow \omega_\mu{}^{\Omega} = S^{-1}\omega_\mu S + \mathrm{i}\,\partial_\mu S^{-1} S \tag{2.12}$$

and the Lagrangian exhibits the invariance property $\mathscr{L}(e, \psi) = \mathscr{L}(e\Omega, S^{-1}\psi)$. Under these circumstances ω_μ and $\omega_\mu{}^{\Omega}$ may be regarded as physically equivalent. However, the ability to find a function S such that (2.11) is true depends on global topological properties of M. Indeed define two $\mathscr{L}_\uparrow$ gauge functions Ω and Ω' to be equivalent if $\Omega^{-1}\Omega'$ possesses a covering map. Then the set of such equivalence classes may be given an abelian group structure and is isomorphic to $H^1(M; Z_2)$.

Thus if the gauge function Ω_h represents a particular non trivial element h in $H^1(M; Z_2)$ the gauge transformation (2.9) and (2.10) *cannot* be compensated by an $SL(2, \mathbb{C})$ transformation of the spinor field and as a result $\omega_\mu{}^{\Omega}$ must be regarded as being physically distinct from ω_μ. Specific calculations of self-energies or vacuum polarisations substantiate this (DeWitt *et al* 1979, Ford 1979, 1980a, b). The term 'twisted spin connection' refers to the objects $\omega_\mu{}^{\Omega}$ and they are classified by elements of the cohomology group $H^1(M; Z_2)$. Note that the twist is relative to a given initial choice of vierbein and hence trivialisation of the tangent bundle. The connection with Milnor's definition (Milnor 1963) of a spin structure is easily made once it is observed that an $SL(2, \mathbb{C})$ bundle over a non compact four-dimensional manifold is necessarily globally trivial (Geroch 1968).

2.3.

It is striking that the topological classes of twisted scalar fields and spin connections are both labelled by the cohomology group $H^1(M; Z_2)$. To understand the significance of this in a supersymmetric model it would clearly be useful if the twist in the spin connection could in some way be transferred to the spinor field.

This may be acheived by considering the vierbein gauged Lagrangian (2.5):

$$\mathscr{L}(e\Omega, \psi) = \det(e)\{\tfrac{1}{2}(\bar{\psi}\gamma_a \nabla_\mu(\omega^\Omega)\psi - \nabla_\mu(\omega^\Omega)\bar{\psi}\gamma_a\psi)e^{b\mu}\Omega_b{}^a - m\bar{\psi}\psi\} \tag{2.13}$$

in which Ω is an $\mathscr{L}_\uparrow$-valued gauge function that cannot be globally lifted to $SL(2, \mathbb{C})$. Cover M with a family $\{U_\alpha\}$ of contractible open sets—this is always possible since M is a manifold. On each U_α the function Ω *can* be lifted to an $SL(2, \mathbb{C})$-valued function $S_{(\alpha)}$ such that, for all x in U_α,

$$\Lambda(S_{(\alpha)}(x)) = \Omega(x) \tag{2.14}$$

and ω^Ω satisfies (2.12). Now define $\psi_{(\alpha)}(x) \equiv S_{(\alpha)}(x)\psi(x)$ and note that on U_α the effects of Ω can be locally gauged away:

$$\mathscr{L}(e\Omega, \psi) = \mathscr{L}(e, \psi_{(\alpha)}). \tag{2.15}$$

However, by the assumption on Ω, $\psi_{(\alpha)}$ cannot be extended to a global spinor field on M. Instead on $U_\alpha \cap U_\beta$ we have $\psi_{(\alpha)}(x) = S_{(\alpha)}(x)S_\beta^{-1}(x)\psi_{(\beta)}(x)$ and $\Lambda(S_{(\alpha)}) = \Lambda(S_{(\beta)})$ which implies $\Lambda(S_{(\alpha)}S_{(\beta)}^{-1}) = \mathbb{1}$. Hence $S_{(\alpha)}S_{(\beta)}^{-1}$ belongs to the Z_2 centre of $SL(2, \mathbb{C})$ and on $U_\alpha \cap U_\beta$

$$\psi_{(\alpha)}(x) = g_{\alpha\beta}(x)\psi_{(\beta)}(x) \tag{2.16}$$

where $g_{\alpha\beta} \equiv S_{(\alpha)}S_{(\beta)}^{-1}$ maps M into Z_2. Clearly $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$, $g_{\alpha\alpha} = \mathbb{1}$ and $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$. Thus $\{g_{\alpha\beta}\}$ are the structure functions of a principal Z_2-bundle over M which in fact corresponds to the element of $H^1(M; Z_2)$ associated earlier with Ω. (This bundle is of course trivial when viewed as an $SL(2, \mathbb{C})$ bundle.) Equation (2.16) shows that the set of functions $\{\psi_{(\alpha)}\}$ defines a cross-section of an associated vector bundle and we shall refer to such sections as twisted spinor fields. It is apparent from (2.15) that, as desired, the spin connection's twist may be absorbed in a twist in the spinor field.

There is another useful way of viewing this construction. The map Ω from M into $\mathscr{L}_\uparrow$ pulls back a Z_2-bundle $\hat{M}$ over M from the Z_2-bundle with projection map Λ from $SL(2, \mathbb{C})$ onto $\mathscr{L}_\uparrow$. In the usual way there exists a bundle map $\hat{S}$ from $\hat{M}$ into $SL(2, \mathbb{C})$, with the commutative diagram

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{S}} & SL(2, \mathbb{C}) \\ \big\downarrow \Pi & & \big\downarrow \Lambda \\ M & \xrightarrow{\Omega} & \mathscr{L}_\uparrow \end{array} \tag{2.17}$$

If ψ is a normal spinor field function from M into $\mathbb{C}^n$, a new field may be defined on $\hat{M}$ by $\hat{\psi}(p) \equiv \hat{S}(p)\psi(\pi(p))$. Since $\hat{S}(pa) = -\hat{S}(p)$ we see that $\hat{\psi}(pa) = -\hat{\psi}(p)$ and hence $\hat{\psi}$ defines a cross-section of the bundle associated with the Z_2-bundle $\hat{M}$. The local representatives of this cross-section are simply the $\psi_{(\alpha)}$ defined above.

2.4.

Let $Z_h[e] \equiv Z[e\Omega_h]$ denote the generating functional for a quantised spinor field with external vierbein and spin connection ω^{Ω_h} where Ω_h is a gauge function representing a particular h in $H^1(M; Z_2)$:

$$Z[e] = \int \delta\bar{\psi}\delta\psi \, \mathrm{e}^{(\mathrm{i}/\hbar)\int_M \mathscr{L}(e,\psi)}. \tag{2.18}$$

Then, as shown in Avis and Isham (1979), an $\mathscr{L}_\uparrow$ gauge invariant (up to an irrelevant overall factor) functional is

$$Z_\chi[e] = \sum_{h \in H^1(M;Z_2)} \chi(h) Z_h[e] \tag{2.19}$$

where χ is a character on $H^1(M; Z_2)$ which thus labels the vacuum state by analogy with the Yang–Mills θ-parameter. This construction plus those in § 2.3 are valid for both complex and real (Majorana) spinors of any dimension. However, a phenomenon arises for complex spinors which was first discussed by Petry (1979) and is worth outlining in the context of the present paper.

Suppose that $H^2(M; Z)$ has no elements of order 2. Then any principal Z_2-bundle over M is trivial if viewed as a $U1$ bundle and there will exist functions λ_α from U_α into $U1$ such that on $U_\alpha \cap U_\beta$ the structure functions of (2.16) obey

$$g_{\alpha\beta}(x) = \lambda_\alpha^{-1}(x)\lambda_\beta(x) \qquad \text{i.e. } \lambda_\alpha S_\alpha = \lambda_\beta S_\beta. \tag{2.20}$$

Define new spinor functions $\chi_{(\alpha)}(x) \equiv \lambda_\alpha(x)\psi_{(\alpha)}(x)$. Then from (2.20) $\chi_{(\alpha)}(x) = \chi_{(\beta)}(x)$ and hence a global spinor χ is defined. In effect we have exploited the fact that (Avis and Isham 1980), on a manifold M where $H^2(M; Z)$ has no two-torsion, the gauge function Ω can be globally lifted to a function from M into $\mathrm{Spin}^c(3, 1) \approx SL(2, \mathbb{C}) \times_{Z_2} U1$. This group was employed in Hawking and Pope (1978), as a means of constructing spinor fields on manifolds that do not admit them in a conventional way, but clearly it also plays a role in understanding the significance of inequivalent spin structures.

On U_α we find

$$\mathscr{L}(e\Omega, \psi) = \det e\{\tfrac{1}{2}\mathrm{i}[\bar{\chi}\gamma_a(\nabla_\mu(\omega) + \lambda_\alpha \partial_\mu \lambda_\alpha^{-1})\chi\, e^{a\mu} + \mathrm{HC}] - m\bar{\chi}\chi\} \tag{2.21}$$

and see that the topological information carried by Ω is absorbed entirely by an effective electromagnetic field $V_\mu^{(\alpha)} \equiv 1/\mathrm{i}\lambda_\alpha\, \partial_\mu \lambda_\alpha^{-1}$. Since $g_{\alpha\beta}^2 = \mathbb{1}$ it follows that $\lambda_\alpha^2 = \lambda_\beta^2$ and hence we obtain a global function ρ from M into U1 by defining the restriction of ρ to U_α to be λ_α^2. Correspondingly $V_\mu \equiv \frac{1}{2}\rho\, \partial_\mu \rho^{-1}/\mathrm{i}$ gives a globally defined field which is just the pullback by ρ of the Cartan Maurer structure form θ on $U1$. The structure equations $\mathrm{d}\theta + \theta \wedge \theta = 0$ show that in the present abelian case θ and hence $V = \rho^*\theta$ are closed forms. However, V is not exact since if it were there would exist a real function α on M such that $\rho(x) = a\ \mathrm{e}^{\mathrm{i}\alpha(x)}$ but the Spin^c (3,1) argument may be developed to show that ρ is not homotopic to a constant and hence cannot be of this exponential form. Thus we see that the spinor twists may be absorbed in an external electromagnetic potential representing an element of $H^1(M; R)$.

This possibility is not present if Majorana spinors are employed as they cannot carry electromagnetic charge and the full topological treatment culminating in (2.19) is appropriate. This leads naturally to the consideration of supersymmetry multiplets which intrinsically contain such spinors.

3. Twisted supermultiplets

3.1.

Simple $N=1$ supergravity contains a vierbein $e_{a\mu}$, a spin-$\frac{3}{2}$ Majorana spinor ψ_μ and, in its minimal form, two auxiliary scalar fields M and N and an auxiliary vector field b_μ (Stelle and West 1978a, Ferrara and van Nieuwenhuizen 1978a). In the notation of Stelle and West (1978a) the Lagrangian is

$$\mathscr{L}_{\mathrm{SG}} = \det(e)\{\tfrac{1}{2}R(e, \omega(e, \psi))/\kappa^2 - \tfrac{1}{3}M^2 - \tfrac{1}{3}N^2 + \tfrac{1}{3}b_\mu b^\mu\} - \tfrac{1}{2}\mathrm{i}\bar{\psi}_\mu\gamma_5\gamma_\nu\nabla_\rho\psi_\kappa\epsilon^{\mu\nu\rho\kappa} \tag{3.1}$$

$$\omega_{\mu ab}(e, \psi) \equiv \omega_{\mu ab}(e) + \tfrac{1}{4}\mathrm{i}\kappa^2\{\bar{\psi}_\mu\gamma_a\psi_b + \bar{\psi}_a\gamma_\mu\psi_b - \bar{\psi}_\mu\gamma_b\psi_a\} \tag{3.2}$$

where $\omega_{\mu ab}(e)$ is the spin connection of (2.8) and R is the scalar curvature.

Under the action of an infinitesimal supersymmetry element with Grassmann parameter ϵ the fields transform as

$$\delta e_{a\mu} = \mathrm{i}\kappa\bar{\epsilon}\gamma_a\psi_\mu \tag{3.3}$$

$$\delta\psi_\mu = (2/\kappa)\nabla_\mu(e, \omega(e, \psi))\epsilon + \gamma_5(b_\mu - \tfrac{1}{3}\gamma_\mu\gamma_\nu b^\nu)\epsilon - \tfrac{1}{3}\gamma_\mu(M + \gamma_5 N)\epsilon \tag{3.4}$$

$$\delta M = -\tfrac{1}{2}\mathrm{i}(\det e)^{-1}\bar{\epsilon}\gamma_\mu\gamma_\nu\gamma_5\nabla_\rho\psi_\kappa\epsilon^{\mu\nu\rho\kappa} - \tfrac{1}{2}\mathrm{i}\kappa\bar{\epsilon}\gamma_5 b^\nu\psi_\nu - \tfrac{1}{2}\mathrm{i}\kappa\bar{\epsilon}\gamma^\nu\psi_\nu M + \tfrac{1}{2}\mathrm{i}\kappa\bar{\epsilon}\gamma_5\gamma^\nu\psi_\nu N \tag{3.5}$$

$$\delta N = -\tfrac{1}{2}\mathrm{i}(\det e)^{-1}\bar{\epsilon}\gamma_5\gamma_\mu\gamma_\nu\gamma_5\nabla_\rho\psi_\kappa\epsilon^{\mu\nu\rho\kappa} + \tfrac{1}{2}\mathrm{i}\kappa\bar{\epsilon}b^\nu\psi_\nu + \tfrac{1}{2}\mathrm{i}\kappa\bar{\epsilon}\gamma^\nu\psi_\nu N - \tfrac{1}{2}\mathrm{i}\kappa\bar{\epsilon}\gamma_5\gamma^\nu\psi_\nu M \tag{3.6}$$

$$\begin{aligned}\delta b_\mu = {} & \tfrac{3}{2}\mathrm{i}(\det e)^{-1}\bar{\epsilon}\gamma_5(g_{\mu\nu} - \tfrac{1}{3}\gamma_\mu\gamma_\nu)\gamma_5\gamma_\alpha\nabla_\beta\psi_\gamma\epsilon^{\nu\alpha\beta\gamma} + \mathrm{i}\kappa\bar{\epsilon}\gamma^\nu b_\nu\psi_\mu \\ & - \tfrac{1}{2}\mathrm{i}\kappa\bar{\epsilon}\gamma^\nu\psi_\nu b_\mu - \tfrac{1}{2}\mathrm{i}\kappa\bar{\psi}_\mu(M + \gamma_5 N)\gamma_5\epsilon - \tfrac{1}{4}\mathrm{i}\kappa(\det e)^{-1}\epsilon_\mu{}^{\alpha\beta\rho}b_\alpha\bar{\epsilon}\gamma_5\gamma_\beta\psi_\rho.\end{aligned} \tag{3.7}$$

Our task is to see what, if any, of these fields can be twisted. The proof of the invariance (up to a four divergence) of $\mathscr{L}_{\mathrm{SG}}$ depends on only the local algebraic properties of the Lagrangian and the group transformations. Hence this invariance will not be disturbed by twisting the fields and the only requirement is to ensure that (3.1)–(3.7) are well defined.

Consider first the Lagrangian $\mathscr{L}_{\mathrm{SG}}$ which we require to be a normal function. The natural starting point is to twist the spin connection or equivalently, as shown in § 2.3, to twist the spinor field ψ_a. The square ϕ^2 of a cross-section ϕ of a real line bundle may be defined as a normal function by equating ϕ^2 restricted to U_α with $\phi^2_{(\alpha)}$. There is consistency on $U_\alpha \cap U_\beta$ since $g_{\alpha\beta}{}^2 = \mathbb{1}$ and hence $\phi^2_{(\alpha)} = \phi^2_{(\beta)}$. This result extends naturally to products like $\bar{\psi}\psi$. In the present context this means that $\bar{\psi}_\mu\gamma_5\gamma_\nu\nabla_\rho\psi_\kappa\epsilon^{\mu\nu\rho\kappa}$ is a normal function even if ψ is twisted provided that, as we shall assume from now on, the vierbein is a standard one. Similarly terms like M^2 and N^2 are admissible whatever line bundles are employed as is $b_\mu b^\mu$ if the obvious concept of a twisted vector field is introduced (a cross-section of the tensor product of the appropriate line bundle with the tangent bundle). Thus as far as the Lagrangian is concerned, any topological configurations for the fields ψ_μ, M, N and b_μ may be assumed.

The supersymmetry transformations, however, impose severe restrictions. Firstly equation (3.3) shows that in order for $e_{a\mu}$ to remain normal the supersymmetry parameter ϵ must itself be twisted with the same degree of twist as ψ_μ. This is consistent with the first term $\nabla_\mu\epsilon$ in $\delta\psi_\mu$ (3.4) whilst the remaining two terms require M, N and b_μ to be normal fields. It may readily be checked that these topological assignments are compatible with the equations for δM, δN and δb_μ. Thus one concludes that in simple supergravity any degree of twist may be carried by the spin-$\frac{3}{2}$ field (and shared by ϵ) but the auxiliary fields must be normal. The results of § 2.4 show that the vacuum states are labelled by the characters of $H^1(M; Z_2)$.

In the full quantisation of the theory it is necessary to introduce ghost fields C^ρ, C^{ab} and C(a complex spinor) corresponding respectively to the supersymmetry, Lorentz and general coordinate gauge transformations. These ghost fields are, together with the dynamical fields, invariant under the BRS transformations (Stelle and West 1978b) and this must be maintained in the topologically non-trivial sectors. An inspection of these transformations shows that this is so provided that C^ρ and C^{ab} are normal fields while C carries the same degree of twist as ψ_μ. The Fadeev–Popov terms in the Lagrangian are fully compatible with this assignment.

3.2.

Let us now consider the more complicated case of a scalar multiplet (A, B, χ, F, G) (Wess and Zumino 1974a, b) in which A, B, F, G are scalars and χ is a Majorana spinor coupled to supergravity. There are various forms of this (Ferrara *et al* 1977, Cremmer and Scherk 1977, Das *et al* 1977, Stelle and West 1978c, Ferrara and van Nieuwenhuizen 1978b) and for convenience we shall adopt the scheme of Stelle and West (1978) which gives a superconformally invariant theory. The Lagrangian is

$$\begin{aligned}\mathscr{L} = &-\det e\{\hat\nabla_a A\hat\nabla_a A + \hat\nabla_a B\hat\nabla_a B + \mathrm{i}\bar\chi\gamma^a\hat\nabla_\mu\chi e^{a\mu} - F^2 - G^2\}\\ &+\tfrac{1}{2}\mathrm{i}\kappa \det e\bar\psi_\mu\gamma_5\gamma^\mu[G+\gamma_5 F - \hat\nabla_a(B+\gamma_5 A)\gamma^a]\chi - \tfrac{2}{3}\kappa \det eN(FB+GA)\\ &+\tfrac{2}{3}\kappa \det eM(GB-FA)\\ &-\tfrac{2}{3}\kappa(\det eb^a + \tfrac{3}{8}\mathrm{i}\kappa\epsilon^{\mu\rho a\tau}\bar\psi_\rho\gamma_\tau\psi_\mu)(B\hat\nabla_a A - A\hat\nabla_a B - \tfrac{1}{2}\mathrm{i}\bar\chi\gamma_5\gamma_a\chi)\\ &-\tfrac{1}{3}\mathrm{i}\kappa\bar\chi(B-\gamma_5 A)(\gamma_5\gamma_\mu\gamma_5\gamma_\nu\nabla_\rho\psi_\kappa\epsilon^{\mu\nu\rho\kappa} + \tfrac{3}{8}\mathrm{i}\kappa^2\epsilon^{\mu\rho a\tau}\psi_a\bar\psi_\rho\gamma_\tau\psi_\mu)\\ &-\tfrac{1}{3}\kappa^2(A^2+B^2)\mathscr{L}_{\mathrm{SG}}\end{aligned}\tag{3.8}$$

where

$$\hat\nabla_a A = \partial_a A - \tfrac{1}{2}\mathrm{i}\kappa\bar\psi_a\chi, \qquad \hat\nabla_a B = \partial_a B - \tfrac{1}{2}\mathrm{i}\kappa\bar\psi_a\gamma_5\chi \tag{3.9}$$

$$\hat\nabla_\mu\chi = [\nabla_\mu(\omega(e,\psi)) - \tfrac{1}{2}\kappa b_\mu\gamma_5]\chi - \tfrac{1}{2}\kappa[\gamma^\nu\hat\nabla_\nu(A+\gamma_5 B) + F + \gamma_5 G]\psi_\mu \tag{3.10}$$

and the supersymmetry transformations are

$$\delta A = \mathrm{i}\bar\epsilon\chi, \qquad \delta B = \mathrm{i}\bar\epsilon\gamma_5\chi \tag{3.11}$$

$$\delta\chi = [F + \gamma_5 G + \gamma^\mu\hat\nabla_\mu(A+\gamma_5 B)]\epsilon \tag{3.12}$$

$$\delta F = \mathrm{i}\bar\epsilon\gamma^\mu\nabla_\mu\chi - \mathrm{i}\kappa\bar\eta\chi, \qquad \delta G = \mathrm{i}\bar\epsilon\gamma_5\gamma^\mu\hat\nabla_\mu\chi + \mathrm{i}\kappa\bar\eta\gamma_5\chi \tag{3.13}$$

where

$$\eta = -\tfrac{1}{3}(M + \gamma_5 N + \gamma^\mu b_\mu\gamma_5)\epsilon. \tag{3.14}$$

One characteristic property of this Lagrangian is the appearance of cubic interactions such as $\bar\psi_\mu\gamma_5\gamma^\mu G\chi$ involving three different fields. By analogy with the assignments in § 3.1, one possible scheme is to make two of these twisted to the same degree and the third one normal. Equations (3.8)–(3.14) give rise to two alternatives. They are (i) A, B, F, G twisted, χ normal and (ii) A, B, F, G normal, χ twisted, all twists to be equal to that of ψ_μ and ϵ.

There is however, a much more general possibility. First recall that if h and h' are two elements in $H^1(M; Z_2)$ with corresponding principal Z_2-bundles ξ_h and $\xi_{h'}$, then $h+h'$ represents the tensor product bundle $\xi_h \otimes \xi_{h'}$. Secondly, let ϕ and ϕ' be cross-sections of real line bundles associated with ξ_h and $\xi_{h'}$ and let $\{\phi_{(\alpha)}\}$ and $\{\phi'_{(\alpha)}\}$ be their local representatives on the contractible open sets $\{U_\alpha\}$. Now on $U_\alpha \cap U_\beta$, as in (2.2),

$$\phi_{(\alpha)}(x) = g_{\alpha\beta}(x)\phi_{(\beta)}(x), \qquad \phi'_{(\alpha)}(x) = g'_{\alpha\beta}(x)\phi'_{(\beta)}(x) \tag{3.15}$$

and so

$$\phi_{(\alpha)}(x)\phi'_{(\alpha)}(x) = g_{\alpha\beta}(x)g'_{\alpha\beta}(x)\phi_{(\beta)}(x)\phi'_{(\beta)}(x). \tag{3.16}$$

However, $g_{\alpha\beta}g'_{\alpha\beta}$ are the structure functions for $\xi_h \otimes \xi'_h$ and hence (3.16) defines a cross-section of this bundle. In other words, the product of sections of two line bundles is a section of the tensor product bundle. Now suppose that A, B and C are twisted scalar fields with twists corresponding to the group elements h, h' and h'' in $H^1(M; Z_2)$. Then ABC is a cross-section corresponding to $h+h'+h''$. The product ABC will be a normal field, and hence a candidate for a term in a Lagrangian, if and only if

$$h+h'+h''=0 \tag{3.17}$$

which, since every element in $H^1(M; Z_2)$ is of order two, is equivalent to

$$h=h'+h'', \qquad h'=h+h'', \qquad h''=h+h'. \tag{3.18}$$

Using this argument in the case of interest, it may be checked that every term in the Lagrangian and field transformations is consistent with the twist assignments h for ϵ and ψ_μ, h' for χ and h'' for A, B, F, G subject to the condition (3.17) that the topological 'charge' flowing into a vertex must sum to zero in $H^1(M; Z_2)$. Thus in this scalar multiplet theory there are two independent twist assignments h and h' with h'' determined by (3.18).

Suppose that in order to probe the local Lorentz invariance and by analogy with (2.18) we keep $e_{a\mu}$ external and construct the generating functional of products of energy momentum tensors:

$$Z_{h,h'}[e] = \int \delta\psi_\mu \delta A \delta B \delta\chi \delta F \delta G \delta M \delta N \delta b_\mu \exp(\mathrm{i}/\hbar) \int_M \mathscr{L} \tag{3.19}$$

which when integrated over the vierbein yields the full quantum gravity theory in the sector (h, h'). Now consider the effects of a gauge transformation $e_{a\mu}(x) \to e_{\alpha\mu}(x)\Omega^a{}_b(x)$ and take as an example the term $L(e, \chi_{h'}) = \bar{\chi}_{h'}\gamma^a\nabla_a(\omega)\chi_{h'}$ where the suffix on χ denotes the twist. Then from (2.15) $L(e, \chi_{h'}) = L(e\Omega_{h'}, \chi)$ where $\Omega_{h'}$ is a particular function representing h' in $H^1(M; Z_2)$ and χ is a normal untwisted field. We have for arbitrary h'' in $H^1(M; Z_2)$

$$L(e\Omega_{h''}, \chi_{h'}) = L(e\Omega_{h''}\Omega_{h'}, \chi) = L(e\Omega_{h'+h''}, \chi) = L(e, \chi_{h'+h''}) \tag{3.20}$$

and similar rules apply to the other terms in the Lagrangian. Thus

$$Z_{h,h'}[e\Omega_{h''}] = Z_{h+h'',h'+h''}[e] \tag{3.21}$$

and hence this generating functional is not Lorentz gauge invariant.

By analogy with (2.19) one can attempt to construct an invariant functional as a weighted sum

$$Z_\chi[e]=\sum_{h,h'\in H^1(M;Z_2)}\chi(h,h')Z_{h,h'}[e] \tag{3.22}$$

and invariance up to a factor will be achieved if the complex function χ satisfies

$$\chi(h+h'',h'+h'')=\chi(h,h')\chi(h'',h''). \tag{3.23}$$

The transformation $(h,h)\to(h+h'',h'+h'')$ may be viewed as an action of the diagonal subgroup $\Delta=\{(h,h)\in H^1(M;Z_2)\times H^1(M;Z_2)\}$ on $H^1(M;Z_2)\times H^1(M;Z_2)$. This action is not transitive and correspondingly the space splits up into a family of orbits. The crucial observation is that the elements of an orbit are permuted amongst themselves under the action of Δ and consequently it is sufficient in (3.22) to sum only over such a subset. More precisely, invariant functionals may be constructed as follows.

First observe that (3.23) implies that χ must be a character on Δ:

$$\chi(h+h',h+h')=\chi(h,h)\chi(h',h'). \tag{3.24}$$

Next choose any orbit O and select some element (h,h') in it. The function χ may, without any loss of physical generality, be normalised such that $\chi(h,h')=1$. Now choose a character α on Δ and *define* χ on O by

$$\chi(h+h'',h'+h'')\equiv\alpha(h'',h''). \tag{3.25}$$

This clearly satisfies (3.23) on O by construction and so we see that the invariant functionals are labelled by an orbit and a character on Δ. The corresponding vacuum states are a sort of analogue of the θ-vacuum arising in Yang–Mills theory (Jackiw and Rebbi 1976, Callan *et al* 1976). In general $H^1(M;Z_2)$ is a product of N cyclic groups of order two and hence there are 2^N orbits and 2^N characters on Δ so that the total number of elementary invariant functionals is 2^{2N}.

A simple example is afforded by the case $H^1(M;Z_2)\approx Z_2$ (for example if M is a cylinder). The two orbits are $O_1=\{(e,e),(a,a)\}$ and $O_2=\{(e,a),(a,e)\}$ and the functionals, invariant up to a factor, are

$$Z_{O_1,\pm}=Z_{(e,e)}\pm Z_{(a,a)},\qquad Z_{O_2,\pm}=Z_{(e,a)}\pm Z_{(a,e)}$$

where the $\pm$ signs correspond to the two possible characters on Δ with $\alpha(a,a)=1$ and $\alpha(a,a)=-1$ respectively.

3.3.

The analysis of § 3.2 can evidently be extended to other versions of the scalar multiplet coupling with similar results. One exception however, is that the interactions $A(A^2+B^2)$ and $\bar\psi_\mu\sigma^{\mu\nu}(A+\mathrm{i}\gamma_5B)^3\psi_\nu$ of Das *et al* (1977) cannot be permitted unless the scalar fields are assigned zero topological twist.

These techniques can also be readily applied to other supermultiplets. For example the Maxwell multiplet Lagrangian is (Ferrara *et al* 1976, Ferrara and van Nieuwenhuizen 1978b, Stelle and West 1978b)

$$\mathscr{L}=\mathscr{L}_{\mathrm{SG}}-\det e\{\tfrac14F^{\mu\nu}F_{\mu\nu}+\tfrac12\mathrm{i}\bar\lambda\gamma^\mu\hat\nabla_\mu\lambda-\tfrac12D^2+\tfrac14\mathrm{i}\kappa\bar\psi_\tau\sigma^{\mu\nu}\gamma^\tau\lambda F_{\mu\nu}\}$$

where

$$\hat{\nabla}_\mu\lambda \equiv (\partial_\mu + \tfrac{1}{2}\omega_{\mu ab}(e, \psi)\sigma^{ab} + \tfrac{1}{2}\kappa b_\mu\gamma_5)\lambda + \tfrac{1}{2}\kappa(\hat{F}_{\rho\kappa}\sigma^{\rho\kappa} - \gamma_5 D)\psi_\mu$$

$$\hat{F}_{\rho\kappa} = F_{\rho\kappa} - \tfrac{1}{2}\mathrm{i}\kappa(\bar{\psi}_\rho\gamma_\kappa\lambda - \bar{\psi}_\kappa\gamma_\rho\lambda)$$

and the supersymmetry transformations are

$$\delta A_\mu = \mathrm{i}\bar{\epsilon}\gamma_\mu\lambda, \qquad \delta\lambda = -\hat{F}_{\mu\nu}\sigma^{\mu\nu}\epsilon + (\det e)^{-1/2}\gamma_5 D\epsilon,$$

$$\delta D = (\det e)^{1/2}\mathrm{i}\bar{\epsilon}\gamma_5\gamma^\mu\hat{\nabla}_\mu\lambda.$$

The principal enunciated in § 3.2 of twist conservation at a vertex applies here also and leads to the twist assignment h for ψ_μ and ϵ, h' for λ and h'' for D and A_μ subject to $h + h' + h'' = 0$. The analysis of the invariant vacuum generating functionals will be the same as in § 3.2.

4. Conclusion

We have seen that the twist structures of scalar fields and spin connections may be meaningfully combined in supermultiplets in a way which is consistent with the supersymmetry transformations. This requires that the infinitesimal group parameter be itself twisted and that the total twist entering a term in the Lagrangian should sum to zero in $H^1(M; Z_2)$. The twist sums on either side of a supersymmetry transformation equation must also balance. An especial role is played by the Majorana nature of the spinor field which makes it impossible to absorb the topological effects in an electromagnetic field. It would be interesting to investigate from this point of view the $N = 2$ supergravity with a gauged central charge (Zachos 1978) in which a minimally coupled electromagnetic field does appear.

The different topological sectors are in general labelled by a pair of elements from $H^1(M; Z_2)$ and different sectors will lead to different results in the quantum field theory. Lorentz gauge invariant functionals may be constructed by taking suitable linear sums. These functionals are labelled by the orbits and characters of the diagonal subgroup and can be used in a complete quantum theory by functionally integrating over the vierbein with a suitable gauge fixing term.

One significant effect of these topological considerations is upon spontaneous symmetry breaking. A twisted scalar field F must necessarily vanish somewhere. Thus statements such as $\langle F\rangle$ = constant are meaningless and also the field F cannot be added as a term in the Lagrangian. Hence spontaneous symmetry breaking is suppressed in topological non trivial sectors. The effects of this on, for example, the classification of stable ground states can be quite dramatic (Avis and Isham 1978).

It should be noted that in this paper we have only considered infinitesimal supersymmetry transformations. However, it is conceivable that new topological effects could be revealed in a study of finite transformations. A start has been made in this direction by R Yates (to appear) who places the spin-$\frac{3}{2}$ gravitino as a connection in a principal bundle rather than a cross-section of a vector bundle as assumed here.

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References

Avis S J 1979 *PhD thesis* London University
Avis S J and Isham C J 1978 *Proc. R. Soc.* A **363** 581
—— 1979 *Nucl. Phys.* B **156** 441
—— 1980 *Commun. Math. Phys.*, **72** 103
Banach R 1979 *The quantum theory of free automorphic fields*, University of Manchester, preprint
Banach R and Dowker J S 1978 *J. Phys. A: Math. Gen.* **11** 2255
—— 1979a *J. Phys. A: Math. Gen.* **12** 2527
—— 1979b *J. Phys. A: Math. Gen.* **12** 2545
Callan C G, Dashen R F and Gross D J 1976 *Phys. Lett.* B **63** 334
Cremmer E and Scherk J 1977 *Phys. Lett.* B **69** 97
Das A, Fischler M and Roček M 1977 *Phys. Lett.* B **69** 186
DeWitt B S, Hart C and Isham C J 1979 in *Themes in contemporary physics* ed S Deser (Amsterdam: North-Holland)
Ferrara S, Gliozzi F, Scherk J and van Nieuwenhuizen P 1976 *Nucl. Phys.* B **117** 333
Ferrara S, Freedman D Z, van Nieuwenhuizen P, Breitenlohner P, Gliozzi F and Scherk J 1977 *Phys. Rev.* D **15** 1013
Ferrara S and van Nieuwenhuizen P 1978a *Phys. Lett.* B **74** 333
—— 1978b *Phys. Lett.* B **76** 404
Ford L 1979 *Twisted spinor fields in flat spacetimes, Proc. 2nd Marcel Grossmann meeting, Trieste 1979* ed B Ruffini
—— 1980a *Phys. Rev.* D **21** 933
—— 1980b *Phys. Rev.* D **21** 949
Geroch R 1968 *J. Math. Phys.* **9** 1739
Hawking S W and Pope C N 1978 *Phys. Lett.* B **73** 42
Husemoller D 1966 *Fibre bundles* (New York: McGraw-Hill)
Isham C J 1978a *Proc. R. Soc.* A **362** 383
—— 1978b *Proc. R. Soc.* A **364** 591
Jackiw R and Rebbi C 1976 *Phys. Rev. Lett.* **37** 372
Kennedy G, Critchley R and Dowker J S 1979 *Finite temperature field theory with boundaries; stress tensor and surface action renormalisation*, University of Manchester, preprint
Milnor J 1963 *Enseign. Math.* **9** 198
Petry H R 1979 *J. Math. Phys.* **20** 231
Stelle K and West 1978a *Phys. Lett.* B **74** 330
—— 1978b *Nucl. Phys.* B **140** 285
—— 1978c *Nucl. Phys.* B **145** 175
Toms D J 1979a *The Casimir effect and topological mass*, University of Toronto, preprint
—— 1979b *Symmetry breaking and mass generation by spacetime topology*, University of Toronto, preprint
Unwin S D 1979 *J. Phys. A: Math. Gen.* **13** L309
—— 1980 *J. Phys. A: Math. Gen.* **13,** 313
Unwin S D and Critchley 1979 *Atomic ground state in multiply connected universes*, University of Manchester, preprint
Wess J and Zumino B 1974a *Nucl. Phys.* B **70** 39
——1974b *Phys. Lett.* B **49** 52
Zachos C K 1978 *Phys. Lett.* B **76** 329